Two-periodic Aztec diamond

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joint work with

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Optimal and Random Point Configurations

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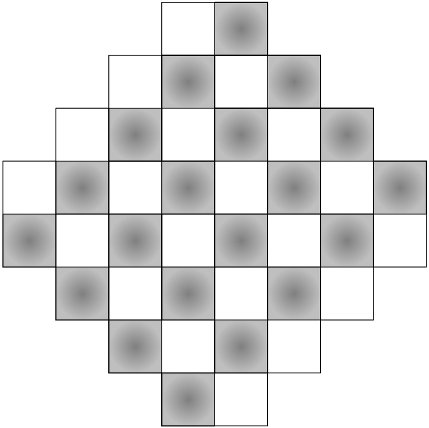
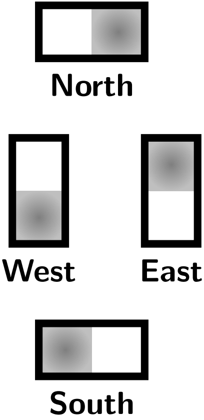
Outline

1. **Aztec diamond**
2. **The model and main result**
3. **Non-intersecting paths**
4. **Matrix Valued Orthogonal Polynomials (MVOP)**
5. **Analysis of RH problem**
6. **Saddle point analysis**
7. **Periodic tilings of a hexagon**

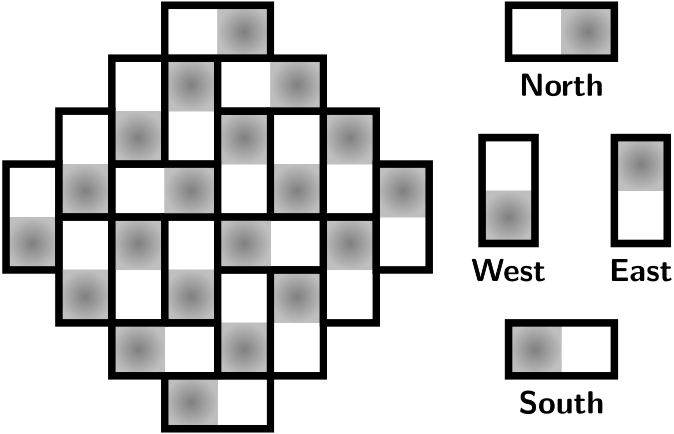
1. Aztec diamond

Aztec diamond



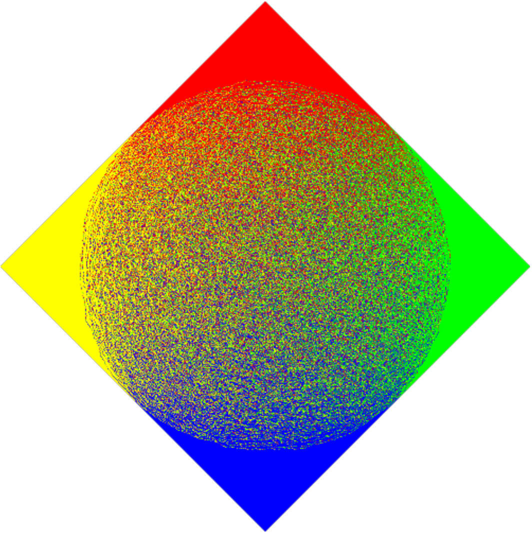


Tiling of an Aztec diamond



- Tiling with 2×1 and 1×2 rectangles (dominos)
- Four types of dominos

Large random tiling



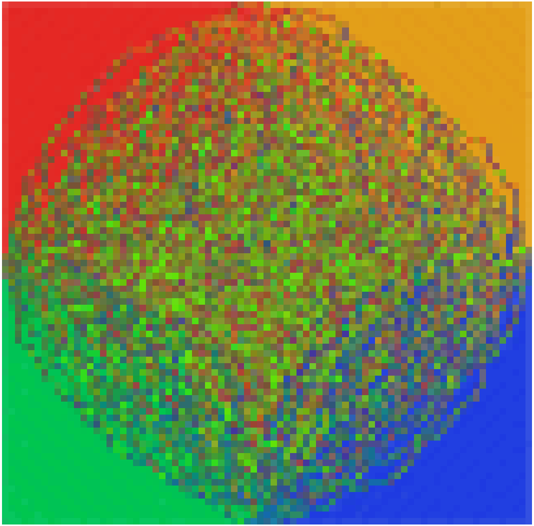
Deterministic pattern near corners
Solid region
or
Frozen region

Disorder in the middle
Liquid region

Boundary curve
Arctic circle

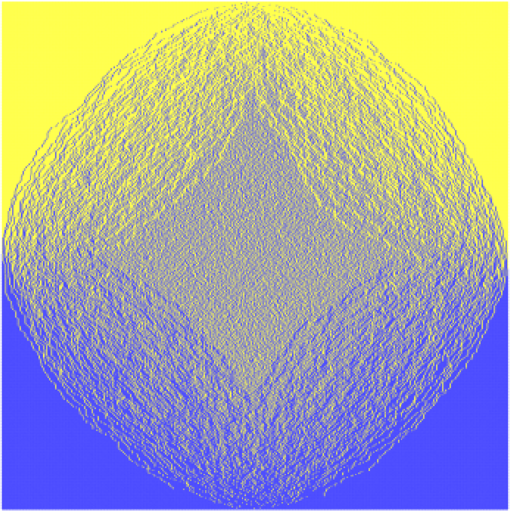
Recent development

- **Two-periodic weighting** Chhita, Johansson (2016) Beffara, Chhita, Johansson (2018 to appear)

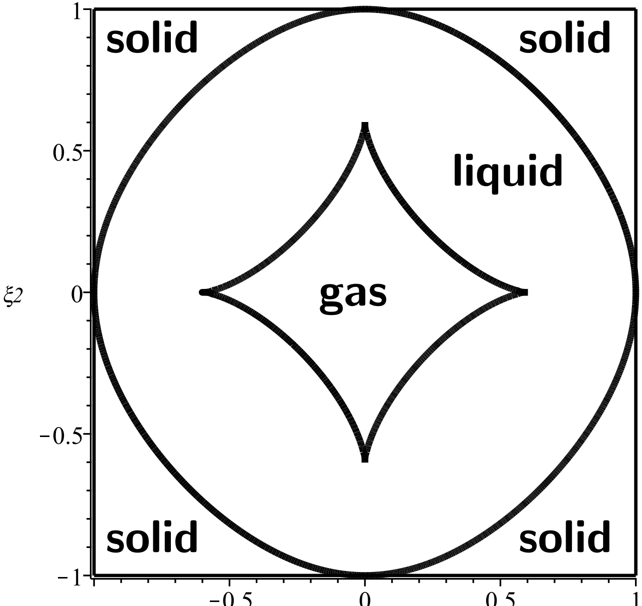


Two-periodic weights

- **A new phase within the liquid region: gas region**

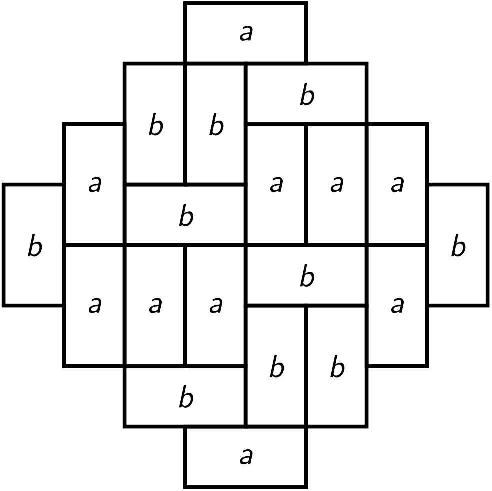


Phase diagram



2. The model and main result

Two periodic weights



Aztec diamond of size $2N$

Weight $w(T)$ of a tiling T is the product of the weights of dominos

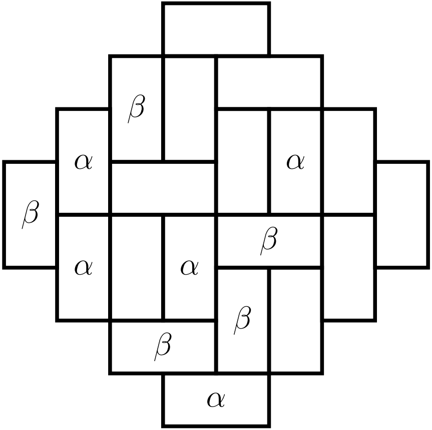
Partition function

$$Z_N = \sum_T w(T)$$

Probability for T

$$\textbf{Prob}(T) = \frac{w(T)}{Z_N}$$

Equivalent weights



$\alpha = a^2$ **and** $\beta = b^2$

North and East dominos have weight 1

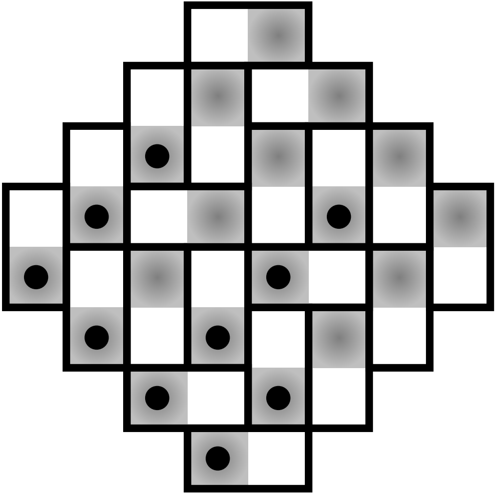
Without loss of generality

$$\alpha\beta = 1$$

and $\alpha \geq 1$

Since North dominos have weight 1, we can transfer the weights to **non-intersecting paths.**

Particles in West and South dominos

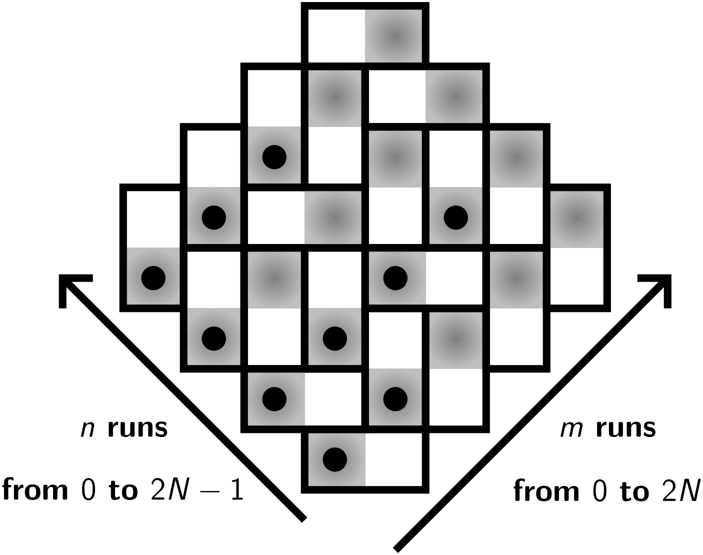


Particles along diagonal lines are interlacing

Positions of particles are random in the two-periodic Aztec diamond. Structure of determinantal point process

- **We found explicit formula for kernel K_N using matrix valued orthogonal polynomials (MVOP).**

Coordinates



Formula for correlation kernel

THEOREM 1 Assume N is even and $m+n$ and $m'+n'$ are even.

$$\begin{pmatrix} K_N(m,n;m',n') & K_N(m,n+1;m',n') \\ K_N(m,n;m',n'+1) & K_N(m,n+1;m',n'+1) \end{pmatrix}$$

$$= -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma_{0,1}} A^{m-m'}(z) z^{\frac{m'-m+n'-n}{2}} \frac{dz}{z} +$$

$$\frac{1}{(2\pi i)^2}\oint_{\gamma_{0,1}} \frac{dz}{z} \oint_{\gamma_1} \frac{dw}{z-w} \frac{z^{\frac{N-m-n}{2}}(z-1)^N}{w^{\frac{N-m'-n'}{2}}(w-1)^N} A^{N-m'}(w) F(w) A^{-N+m}(z)$$

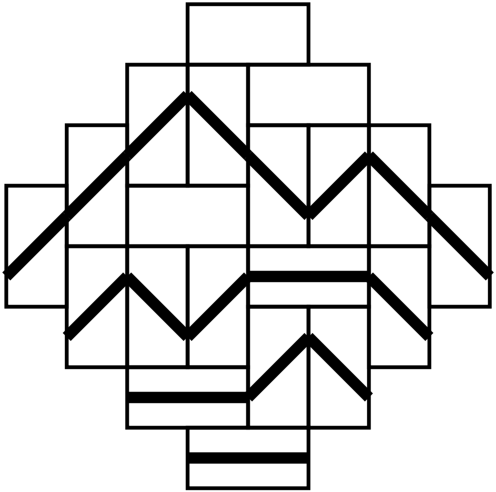
where

$$A(z) = \frac{1}{z-1}\begin{pmatrix} 2\alpha z & \alpha(z+1) \\ \beta z(z+1) & 2\beta z \end{pmatrix}$$

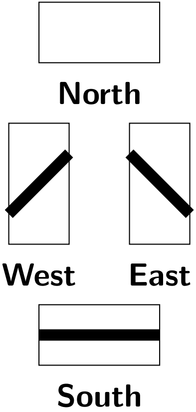
$$F(z) = \frac{1}{2} I_2 + \frac{1}{2\sqrt{z(z+\alpha^2)(z+\beta^2)}}\begin{pmatrix} (\alpha-\beta)z & \alpha(z+1) \\ \beta z(z+1) & -(\alpha-\beta)z \end{pmatrix}$$

3. Non-intersecting paths

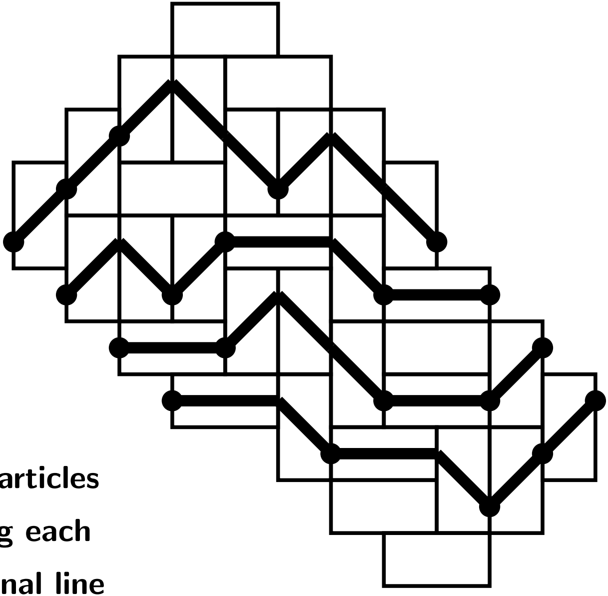
Non-intersecting paths



Line segments on West, East and South dominos



Double Aztec diamond



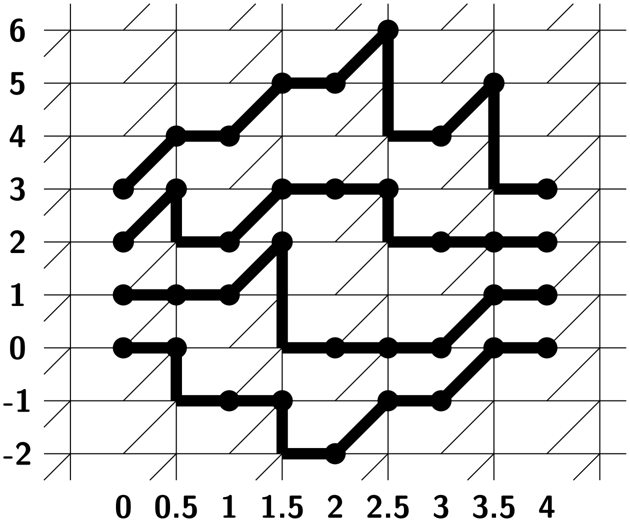
$2N$ **particles**

along each

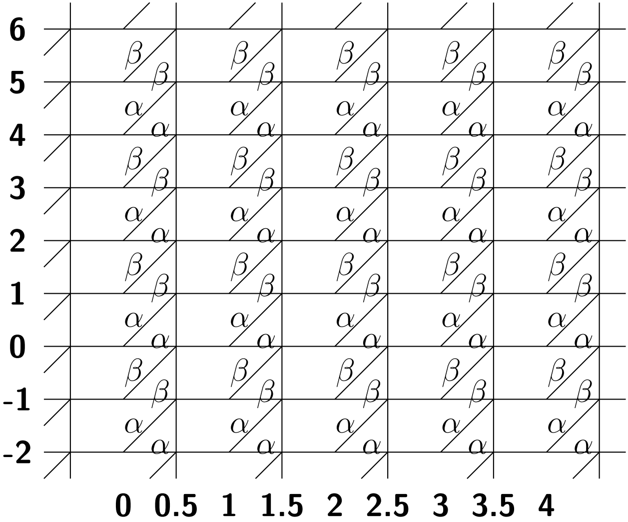
diagonal line

Non-intersecting paths on a graph

Paths are transformed to fit on a graph



Weights on the graph



Weights on non-intersecting paths

Any tiling of double Aztec diamond is equivalent to system $(P_0, \ldots, P_{2N-1})$ of $2N$ non-intersecting paths

- P_j **is path on the graph from $(0, j)$ to $(2N, j)$,**
- P_i **is vertex disjoint from P_j if $i \neq j$.**

Transitions and LGV theorem

There are $2N+1$ **levels,** $0, 1, \ldots, 2N$**.**

- **Transition from level** m **to level** $m' > m$

$$T_{m,m'}(x,y) = \sum_{P:(m,x)\to(m',y)} w(P), \qquad x,y \in \mathbb{Z}$$

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Lindström-Gessel-Viennot theorem

Probability that paths at level m **are at positions** $x_0^{(m)} < x_1^{(m)} < \cdots < x_{2N-1}^{(m)}$**:**

$$\frac{1}{Z_N} \det\left[T_{0,m}(i, x_k^{(m)})\right]_{i,k=0}^{2N-1} \cdot \det\left[T_{m,2N}(x_k^{(m)}, j)\right]_{k,j=0}^{2N-1}$$

Lindström (1973)
Gessel-Viennot (1985)

Determinantal point process

Corollary: The positions at level m are determinantal with kernel

$$K_{N,m}(x,y) = \sum_{i,j=0}^{2N-1} T_{0,m}(i,x) \left[G^{-t}\right]_{i,j} T_{m,2N}(y,j)$$

where $\quad G = [T_{0,2N}(i,j)]_{i,j=0}^{2N-1}$

- **Multi-level extension is known as Eynard-Mehta theorem.**

Block Toeplitz matrices

In our case: Transition matrices are 2 periodic

$$T(x+2,y+2)=T(x,y)$$

- **Block Toeplitz matrices, infinite in both directions,**

  with block symbol $\quad A(z)=\sum_{j=-\infty}^{\infty} B_j z^j$

$$\textbf{if } T=\begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & B_0 & B_1 & \ddots & \\ \ddots & B_{-1} & B_0 & B_1 & \ddots \\ & \ddots & B_{-1} & B_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Double contour integral formula

THEOREM 2: **Suppose transition matrices are 2-periodic. Then**

$$\begin{pmatrix} K_{N,m}(2x,2y) & K_{N,m}(2x+1,2y) \\ K_{N,m}(2x,2y+1) & K_{N,m}(2x+1,2y+1) \end{pmatrix}$$
$$= \frac{1}{(2\pi i)^2} \oint_{\gamma} \oint_{\gamma} A_{m,2N}(w) R_N(w,z) A_{0,m}(z) \frac{w^y}{z^{x+1} w^N} dz dw$$

- $A_{m,2N}$ **and** $A_{0,m}$ **are block symbols for the transition matrices** $T_{m,2N}$ **and** $T_{0,m}$**.**
- $R_N(w,z)$ **is a reproducing kernel for matrix valued polynomials.**

4. Matrix Valued Orthogonal Polynomials (MVOP)

MVOP

- **Matrix valued polynomial of degree j,**

$$P_j(z) = \sum_{i=0}^{j} C_i z^i$$

each C_i **is** $d \times d$ **matrix,** $\det C_j \neq 0$

- $W(z)$ **is** $d \times d$ **matrix valued weight**

- **Orthogonality**

$$\frac{1}{2\pi i} \oint_\gamma P_j(z) W(z) P_k^t(z)\, dz = H_j \delta_{j,k}$$

Reproducing kernel

$$R_N(w,z) = \sum_{j=0}^{N-1} P_j^t(w) H_j^{-1} P_j(z)$$

is reproducing kernel for matrix polynomials of degree $\leq N-1$

- **If Q has degree $\leq N-1$, then**

$$\frac{1}{2\pi i} \oint_\gamma Q(w) W(w) R_N(w,z)\, dw = Q(z)$$

- **There is a Christoffel-Darboux formula for R_N and a Riemann Hilbert problem**

Riemann-Hilbert problem

$Y : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2d \times 2d}$ **satisfies**

- Y **is analytic,**
- $Y_+ = Y_- \begin{pmatrix} I_d & W \\ 0_d & I_d \end{pmatrix}$ **on** γ**,**
- $Y(z) = (I_{2d} + O(z^{-1})) \begin{pmatrix} z^N I_d & 0_d \\ 0_d & z^{-N} I_d \end{pmatrix}$ **as** $z \to \infty$**.**

Grünbaum, de la Iglesia, Martínez-Finkelshtein (2011)

Solution of RH problem

Unique solution (provided P_N uniquely exists) is

$$Y(z) = \begin{pmatrix} P_N(z) & \frac{1}{2\pi i}\oint_\gamma \frac{P_N(s)W(s)}{s-z}ds \\ Q_{N-1}(z) & \frac{1}{2\pi i}\oint_\gamma \frac{Q_{N-1}(s)W(s)}{s-z}ds \end{pmatrix}$$

where P_N is monic MVOP of degree N and $Q_{N-1} = -H_{N-1}^{-1}P_{N-1}$ has degree $N-1$

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Christoffel Darboux formula

$$R_N(w,z) = \frac{1}{z-w}\begin{pmatrix} 0_d & I_d \end{pmatrix} Y^{-1}(w)Y(z)\begin{pmatrix} I_d \\ 0_d \end{pmatrix}$$

Delvaux (2010)

Our case of interest

- **Weight matrix in special case of two periodic Aztec diamond is $W^N(z)$, with**

$$W(z) = \frac{1}{(z-1)^2} \begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

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Scalar valued analogue

- **Weight $\left(\frac{z+1}{z-1}\right)^N$ on circle around $z=1$ and OPs are Jacobi polynomials $P_j^{(-N,N)}(z)$ with nonstandard parameters**

5. Analysis of RH problem

Surprise

Steepest descent analysis of RH problem leads to **explicit formula**

- **RH problem is solved in terms of contour integrals.**
- **For example: MVOP is**

$$P_N(z) = (z-1)^N W_\infty^{N/2} W^{-N/2}(z), \qquad \textbf{if } N \textbf{ is even.}$$

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It leads to proof of THEOREM 1

6. Saddle point analysis

Asymptotic analysis

Saddle point analysis on the double contour integral

$$\frac{1}{(2\pi i)^2}\oint_{\gamma_{0,1}}\frac{dz}{z}\oint_{\gamma_1}\frac{dw}{z-w}\frac{z^{\frac{N-2x}{2}}(z-1)^N}{w^{\frac{N-2y}{2}}(w-1)^N}A^{N-m}(w)F(w)A^{-N+m}(z)$$

when $N \to \infty$

- m, x, y **scale with** N **in such a way that**

$$m \approx (1+\xi_1)N, \qquad x, y \approx (1+\tfrac{\xi_1+\xi_2}{2})N$$

- **Saddle points are critical points of**

$$2\log(z-1) - (1+\xi_2)\log z + \xi_1 \log\lambda(z)$$

where $\lambda(z)$ **is an eigenvalue of** $W(z) = \dfrac{A^2(z)}{z}$**.**

Saddle point analysis

Let $-1 < \xi_1, \xi_2 < 1$. There are always four saddle points, depending on ξ_1, ξ_2, and they lie on the Riemann surface for

$$y^2 = z(z + \alpha^2)(z + \beta^2) \qquad \textbf{(genus one)}$$

with branch points $-\alpha^2 < -\beta^2 < 0$ and infinity.

- **At least two saddles are in $z \in [-\alpha^2, -\beta^2]$.**

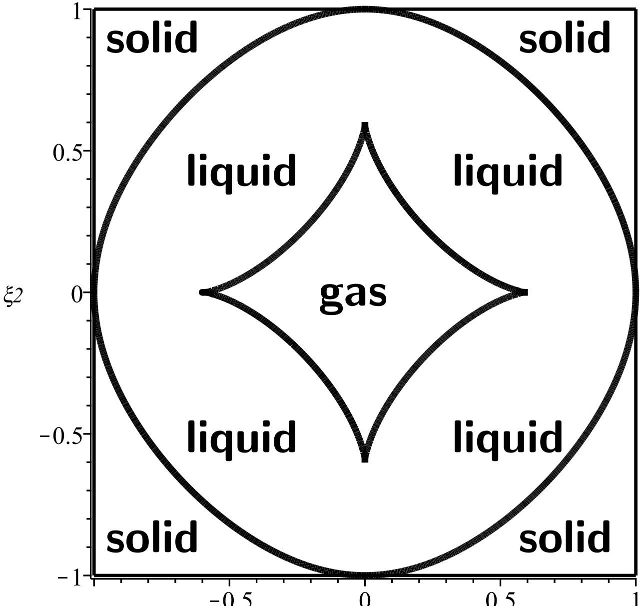
Classification of phases

Location of other two saddles determines the phase.

- **Two saddles are in $[0, \infty)$: solid phase**
- **Two saddles are in $\mathbb{C} \setminus ([-\alpha^2, -\beta^2] \cup [0, \infty))$: liquid phase**
- **All four saddles are in $[-\alpha^2, -\beta^2]$: gas phase**

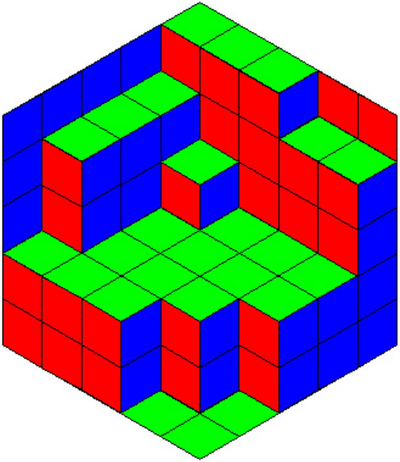
Transitions between phases occur when saddles coalesce.

Phase diagram



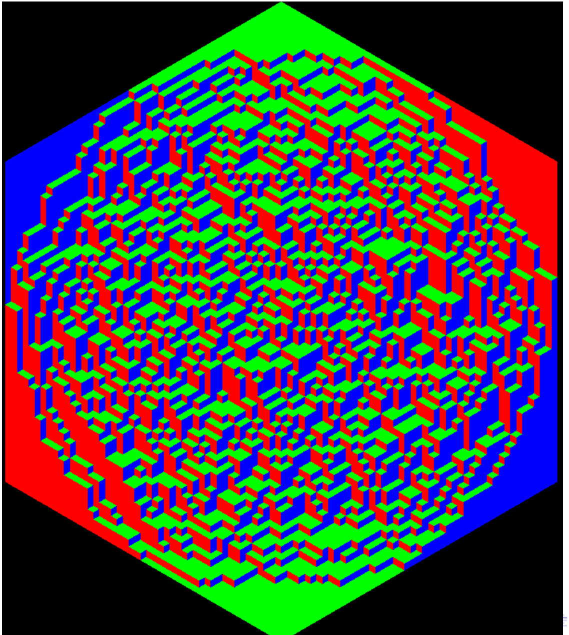
7. Periodic tilings of a hexagon

Tiling of a hexagon

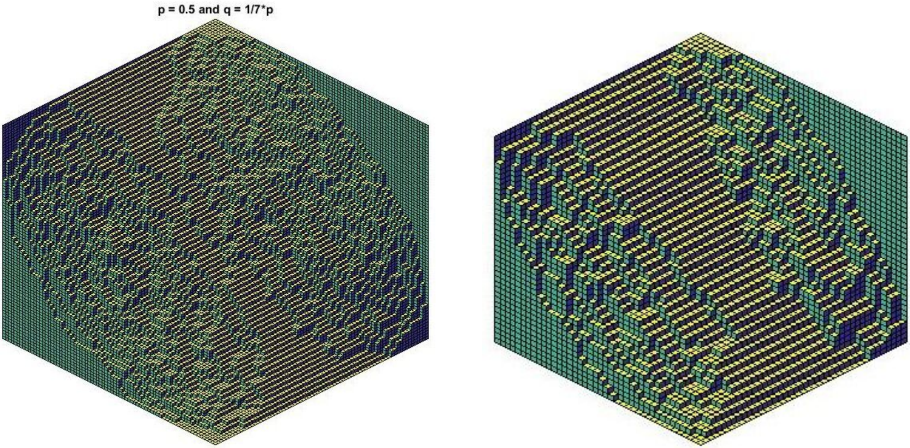


- **Lozenge tiling of a regular hexagon**
- **Also admits a non-intersecting path formulation**

Large random tiling



Two periodic tiling of a hexagon



- **Ongoing work with**
  Charlier, Duits, and Lenells

Thanks

Thank you for your attention